



TITLE:

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AUTHOR(S):

ITAI, Masanori

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# Basic properties of definable isomorphisms of quantum 2-tori $T_q^2(\mathbb{C})$

Masanori ITAI

Department of Mathematical Sciences  
Tokai University, Hiratsuka, Japan

## Abstract

In [5] we constructed quantum 2-tori and studied their first-order theories. Here we discuss basic properties of the definable isomorphisms between quantum 2-tori  $T_q^2(\mathbb{C})$  and show that  $SL_2(\mathbb{Z})$  is the set of definable isomorphisms of them.

## 1 Definable isomorphisms

Let  $\theta \in \mathbb{R}$  be a transcendental element and put  $q = \exp(2\pi i\theta)$  where  $i = \sqrt{-1}$ . Let also  $\Gamma_\theta$  be an infinite multiplicative group generated by  $q$ . We denote  $\mathcal{A}_\theta$  the non-commutative algebra  $\mathcal{O}_q((\mathbb{C}^\times)^2)$  with generators written as  $U, U^{-1}, V, V^{-1}$  satisfying

$$VU = qUV.$$

For each pair  $(u, v) \in \mathbb{C}^* \times \mathbb{C}^* / \Gamma_\theta$ , we construct two  $\mathcal{A}_\theta$ -modules  $M_{|u,v\rangle}$  and  $M_{\langle v,u|}$ .

The module  $M_{|u,v\rangle}$  generated by elements labeled  $\{\mathbf{u}(\gamma u, v) : \gamma \in \Gamma_\theta\}$  satisfies

$$\begin{aligned} U &: \mathbf{u}(\gamma u, v) \mapsto \gamma u \mathbf{u}(\gamma u, v), \\ V &: \mathbf{u}(\gamma u, v) \mapsto v \mathbf{u}(q^{-1} \gamma u, v). \end{aligned} \tag{1}$$

We also define the module  $M_{\langle v,u|}$  generated by elements labeled  $\{\mathbf{v}(\gamma v, u) : \gamma \in \Gamma_\theta\}$  satisfying

$$\begin{aligned} U &: \mathbf{v}(\gamma v, u) \mapsto u \mathbf{v}(q \gamma v, u), \\ V &: \mathbf{v}(\gamma v, u) \mapsto \gamma v \mathbf{v}(\gamma v, u). \end{aligned} \tag{2}$$

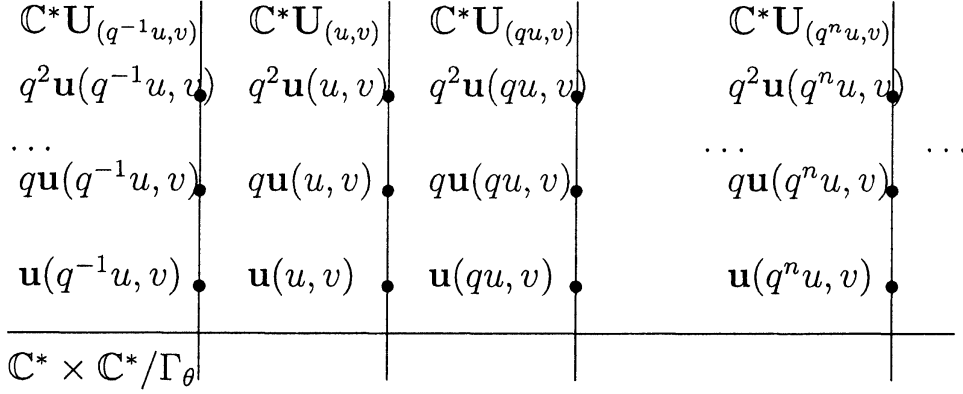


Figure 1:  $\Gamma_\theta$ -bundle over  $\mathbb{C} \times \mathbb{C}/\Gamma_\theta$  inside an ambient  $\mathbb{C}$ -module

Notice that we have

$$\begin{aligned} U^{-1} &: \mathbf{u}(\gamma u, v) \mapsto \gamma^{-1} u^{-1} \mathbf{u}(\gamma u, v), \\ V^{-1} &: \mathbf{u}(\gamma u, v) \mapsto v^{-1} \mathbf{u}(q\gamma u, v), \end{aligned} \quad (3)$$

and

$$\begin{aligned} U^{-1} &: \mathbf{v}(\gamma v, u) \mapsto u^{-1} \mathbf{v}(q^{-1} \gamma v, u), \\ V^{-1} &: \mathbf{v}(\gamma v, u) \mapsto \gamma^{-1} v^{-1} \mathbf{v}(\gamma v, u). \end{aligned} \quad (4)$$

Now let  $\phi: \mathbb{C}^*/\Gamma_\theta \rightarrow \mathbb{C}^*$ . Put  $\Phi = \text{ran}(\phi)$ . Set

$$\begin{aligned} \Gamma_\theta \cdot \mathbf{u}(u, v) &:= \{\gamma \mathbf{u}(u, v) : \gamma \in \Gamma_\theta\}, \\ \mathbf{U}_{\langle u, v \rangle} &:= \bigcup_{\gamma \in \Gamma_\theta} \Gamma_\theta \cdot \mathbf{u}(\gamma u, v) = \{\gamma_1 \cdot \mathbf{u}(\gamma_2 u, v) : \gamma_1, \gamma_2 \in \Gamma_\theta\}, \\ \mathbf{U}_\phi &:= \bigcup_{\langle u, v \rangle \in \Phi^2} \mathbf{U}_{\langle u, v \rangle} \\ &= \{\gamma_1 \cdot \mathbf{u}(\gamma_2 u, v) : \langle u, v \rangle \in \Phi^2, \gamma_1, \gamma_2 \in \Gamma_\theta\}. \end{aligned} \quad (5)$$

We call  $\Gamma_\theta \cdot \mathbf{u}(u, v)$  a  $\Gamma_\theta$ -set over  $\mathbf{u}(u, v)$ , and  $\mathbf{U}_\phi$  a  $\Gamma_\theta$ -bundle over  $\mathbb{C}^* \times \mathbb{C}^*/\Gamma_\theta$ . Notice that  $\mathbf{U}_\phi$  is a subset of the  $\mathcal{A}_\theta$ -module

$$\bigcup_{\langle u, v \rangle} M_{|u, v\rangle}.$$

Similarly a  $\Gamma_\theta$ -set  $\Gamma_\theta \cdot \mathbf{v}(v, u)$  over  $\mathbf{v}(v, u)$ , and a  $\Gamma_\theta$ -bundle  $\mathbf{V}_\phi$  over  $\mathbb{C}^*/\Gamma_\theta \times \mathbb{C}^*$  is defined. As before the  $\Gamma_\theta$ -bundle  $\mathbf{V}_\phi$  is a subset of the  $\mathcal{A}_\theta$ -module

$$\bigcup_{\langle v, u \rangle} M_{\langle v, u|}.$$

**Definition 1** Given  $\theta$  and  $q = \exp(2\pi i\theta)$ . Let  $\phi : \mathbb{C}^*/\Gamma_\theta \rightarrow \mathbb{C}^*$ . We call the structure  $(U_\phi, V_\phi, \mathbb{C})$  with actions  $U$  and  $V$  satisfying (1) and (2) a **quantum 2-torus**  $T_q^2(\mathbb{C})$  over the field of complex numbers  $\mathbb{C}$ . We denote  $T_q^2(\mathbb{C})$  as  $T_\theta$  in this note.

Notice that the structure of the quantum 2-torus  $T_\theta$  is essentially decided by  $\theta$  (in fact  $q = e^{2\pi i\theta}$ ). More precisely  $\Gamma_q = q^{\mathbb{Z}} = \Gamma_\theta = e^{2\pi i\theta\mathbb{Z}} = \langle e^{2\pi i\theta} \rangle$  is the key ingredient of  $\Gamma$ -bundles.

**Remark 2** It is clear that we have  $\Gamma_{\theta\pm 1} = \Gamma_\theta$ , hence for any  $n \in \mathbb{Z}$  we have  $\Gamma_{\theta+n} = \Gamma_\theta$ . It is also clear that  $\Gamma_{-\theta} = \Gamma_\theta$ . For  $n \in \mathbb{Z}$ , it is easy to see that  $\Gamma_{n\theta}$  and  $\Gamma_\theta$  are both infinite multiplicative groups generated by single elements hence isomorphic as such groups.

**Definition 3** Given  $\theta_1, \theta_2 \in \mathbb{R}$ , both transcendental. We say that the quantum 2-tori  $T_{\theta_1}$  and  $T_{\theta_2}$  are definably isomorphic in the structure  $(\mathbb{C}, +, \cdot, x^\theta)$  if there is an definable  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(\theta_1) = \theta_2$  and

1.  $f$  induces a definable isomorphism between  $\mathbb{C}^* \times \mathbb{C}^*/\Gamma_{\theta_1}$  and  $\mathbb{C}^* \times \mathbb{C}^*/\Gamma_{\theta_2}$ ,
2.  $f$  induces a definable isomorphism between  $\Gamma_{\theta_1}$ -bundle over  $\mathbb{C}^* \times \mathbb{C}^*/\Gamma_{\theta_1}$  and  $\Gamma_{\theta_2}$ -bundle over  $\mathbb{C}^* \times \mathbb{C}^*/\Gamma_{\theta_2}$

**Proposition 4** From the definition of definable isomorphism of quantum 2-torus, we see that there are three cases for quantum 2-tori to be definably isomorphic:

1.  $\Gamma_{\theta_1} = \Gamma_{\theta_2}$ , or
2.  $\Gamma_{\theta_1}$  and  $\Gamma_{\theta_2}$  are isomorphic, or
3.  $\mathbb{C}/\Gamma_{\theta_1}$  is definably isomorphic to  $\mathbb{C}/\Gamma_{\theta_2}$ , more precisely there is a definable one-to-one correspondence between the cosets of  $\Gamma_{\theta_1}$  and the cosets of  $\Gamma_{\theta_2}$ .

**Remark 5** Our definition of two quantum 2-tori being isomorphic is essentially the same as the notion of Morita equivalence. The idea of using the Morita equivalence as the definition of isomorphism of quantum tori comes from Manin's argument.

We investigate here the structure of the set  $\text{DefIso}(T_{\theta_1}, T_{\theta_2})$  of all definable isomorphisms  $f$  between quantum tori  $T_{\theta_1}$  and  $T_{\theta_2}$ .

**Proposition 6** *It is clear that from Remark 2, we have (1)  $T_{-\theta}$  and  $T_\theta$  are definably isomorphic and (2)  $T_{\theta+n}$  and  $T_\theta$  are definably isomorphic for  $n \in \mathbb{Z}$ . It is not hard to see that  $T_{n\theta}$  and  $T_\theta$  are definably isomorphic since  $\Gamma_{n\theta}$  and  $\Gamma_\theta$  are isomorphic.*

### 1.1 $x^\theta$ is a definable isomorphism between $T_{\frac{1}{\theta}}$ and $T_\theta$

Here we show that the function  $x^\theta$  gives rise to a definable isomorphism in the sense of Definition 3 between quantum tori  $T_{\frac{1}{\theta}}$  and  $T_\theta$ .

Recall that  $\Gamma_\theta$  is an infinite multiplicative group generated by  $q = \exp(2\pi i\theta)$ .

Recall that over the complex numbers we have

$$x^\theta = e^{\theta \operatorname{Log} x},$$

where  $\operatorname{Log} x = \ln x + 2\pi i\mathbb{Z}$ .

**Remark 7**  $x^\theta$  is not a function but a multivalued operation.

First observation to make is the following:

**Claim** Suppose  $\Gamma_{\theta^{-1}} = \{q_1^k : q_1 = \exp(2\pi i\theta^{-1}), k \in \mathbb{Z}\}$ , and  $\Gamma_\theta = \{q_2^m : q_2 = \exp(2\pi i\theta), m \in \mathbb{Z}\}$ . Then  $x^\theta$  sends a coset of  $\Gamma_{\frac{1}{\theta}}$  to a coset of  $\Gamma_\theta$  as follows;

$$\begin{aligned} x^\theta : a \cdot \exp(2\pi i\theta^{-1}k) &\mapsto \exp\left(\theta(\ln(a \exp(2\pi i\theta^{-1}k)) + 2\pi im)\right) \\ &= \exp\left(\theta(\ln a + 2\pi i\theta^{-1}k + 2\pi im)\right) \\ &= \exp\left(\theta(\ln a + 2\pi im) + 2\pi ik\right) \\ &= \exp(\theta(\ln a + 2\pi im)) \cdot \exp(2\pi ik) \\ &= \exp(\theta \ln a) \cdot \exp(2\pi i\theta m) \\ &= a^\theta \cdot \exp(2\pi i\theta m) \\ &\in a^\theta \Gamma_\theta \end{aligned}$$

On the other hand, for given  $a^\theta \cdot \exp(2\pi i\theta m) \in a^\theta \cdot \Gamma_\theta$  we have  $a \cdot \exp(2\pi i\theta^{-1}m) \in a \cdot \Gamma_{\frac{1}{\theta}}$  and

$$\begin{aligned} x^\theta : a \cdot \exp(2\pi i\theta^{-1}m) &\mapsto \exp\left(\theta(\ln(a \exp(2\pi i\theta^{-1}m)) + 2\pi il)\right) \\ &= a^\theta \cdot \exp(2\pi i\theta l) \\ &\in a^\theta \cdot \Gamma_\theta \end{aligned}$$

These computations show that  $x^\theta$  gives a one-to-one correspondence between a coset of  $\Gamma_{\frac{1}{\theta}}$  and a coset of  $\Gamma_\theta$ . From this we see that  $x^\theta$  defines a definable bijection between  $\mathbb{C}^* \times \mathbb{C}^* / \Gamma_{\frac{1}{\theta}}$  and  $\mathbb{C}^* \times \mathbb{C}^* / \Gamma_\theta$ .

Recall that the module  $M_{|u,v\rangle, \frac{1}{\theta}}$  is generated by elements labeled  $\{\mathbf{u}(\gamma^{\frac{1}{\theta}} u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) : \gamma \in \Gamma_\theta\}$ . Set first

$$x^\theta : \mathbf{u}(\gamma^{\frac{1}{\theta}} u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) \mapsto \mathbf{u}(\gamma u, v)$$

Then the operators  $U, V$  act as follows:

$$\begin{aligned} U &: \mathbf{u}(\gamma^{\frac{1}{\theta}} u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) \mapsto \gamma^{\frac{1}{\theta}} u^{\frac{1}{\theta}} \mathbf{u}(\gamma^{\frac{1}{\theta}} u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}), \\ V &: \mathbf{u}(\gamma^{\frac{1}{\theta}} u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) \mapsto v^{\frac{1}{\theta}} \mathbf{u}(q^{-1} \gamma^{\frac{1}{\theta}} u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}), \end{aligned} \tag{6}$$

where  $q = e^{2\pi i \theta}$ .

Hence we have the following diagrams:

$$\begin{array}{ccc} \mathbf{u}(\gamma^{\frac{1}{\theta}} u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) & \xrightarrow{x^\theta} & \mathbf{u}(\gamma u, v) \\ \downarrow U & \circlearrowleft & \downarrow U \\ \gamma^{\frac{1}{\theta}} u^{\frac{1}{\theta}} \mathbf{u}(\gamma^{\frac{1}{\theta}} u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) & \xrightarrow{x^\theta} & \gamma u \mathbf{u}(\gamma u, v) \end{array}$$

and

$$\begin{array}{ccc} \mathbf{u}(\gamma^{\frac{1}{\theta}} u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) & \xrightarrow{x^\theta} & \mathbf{u}(\gamma u, v) \\ \downarrow V & \circlearrowleft & \downarrow V \\ v^{\frac{1}{\theta}} \mathbf{u}(q^{-1} \gamma^{\frac{1}{\theta}} u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) & \xrightarrow{x^\theta} & v \mathbf{u}(q^{-1} \gamma u, v) \end{array}$$

These diagrams tell us that we have a definable correspondence of vectors in  $\Gamma_{\frac{1}{\theta}}$ -set and  $\Gamma_\theta$ -set and vectors in  $\Gamma_{\frac{1}{\theta}}$ -bundle and  $\Gamma_\theta$ -and, eventually vectors in line-bundles of both sides. Observe that  $x^\theta$  preserves actions of  $U, V$ .

**Proposition 8** *In the structure  $(\mathbb{C}, +, \cdot, x^\theta)$ , by sending  $\frac{1}{\theta}$  to  $\theta$ , we construct a definable isomorphism  $f$  between  $\Gamma_{\frac{1}{\theta}}$  and  $\Gamma_\theta$  which give rise to the definable*

$$f : \mathbb{C}^* / \Gamma_{\frac{1}{\theta}} \xrightarrow{x^\theta} \mathbb{C} / \Gamma_\theta.$$

*Therefore the quantum 2-tori  $T_{\frac{1}{\theta}}$  and  $T_\theta$  are definably isomorphic.*

## 1.2 $\mathrm{SL}_2(\mathbb{Z})$ is the group of definable isomorphisms

We have shown that  $\Gamma_\theta$  and  $\Gamma_{\frac{1}{\theta}}$  define isomorphic quantum 2-tori. Since  $\Gamma_{\theta+1} = \Gamma_\theta$ , it is reasonable to have the following statement.

**Proposition 9**  *$\mathrm{SL}_2(\mathbb{Z})$  is the set of all definable isomorphisms. More precisely, for any  $f \in \mathrm{SL}_2(\mathbb{Z})$ ,  $\Gamma_\theta$  and  $\Gamma_{f(\theta)}$  define isomorphic quantum 2-tori.*

**Proof:** 1) first show that for any  $f \in \mathrm{SL}_2(\mathbb{Z})$ ,  $\Gamma_\theta$  and  $\Gamma_{f(\theta)}$  give rise to isomorphic quantum 2-tori. Consider two functions

$$\begin{aligned} f_S : \theta &\mapsto -\frac{1}{\theta}, \\ f_T : \theta &\mapsto \theta + 1. \end{aligned}$$

View both  $f_S$  and  $f_T$  as Möbius transformations;

$$\theta \mapsto \frac{a\theta + b}{c\theta + d},$$

which is identified with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then  $f_S$  corresponds to  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = S$ , and  $f_T$  corresponds to  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = T$ . Then  $S, T \in \mathrm{SL}_2(\mathbb{Z})$  and  $S$  and  $T$  generate  $\mathrm{SL}_2(\mathbb{Z})$ , i.e.,  $\langle S, T \rangle = \mathrm{SL}_2(\mathbb{Z})$ . From the argument in the previous subsection we have that  $\mathrm{SL}_2(\mathbb{Z})$  is a subgroup of the group of definable isomorphisms.

We still need to show that any definable isomorphism in fact belongs to  $\mathrm{SL}_2(\mathbb{Z})$ . For this we have

**Lemma 10** *Suppose  $T_\theta$  and  $T_{\theta_1}$  are definably isomorphic quantum tori in  $(\mathbb{C}, +, \cdot, x^\theta)$ . Then there is a  $f \in \langle S, T \rangle$  such that  $\theta_1 = f(\theta)$ .*

**Proof:** Recall Proposition 4. From the fact that  $\theta_1$  being definable in the structure  $(\mathbb{C}, +, \cdot, x^\theta)$ , there are three cases to consider;

1.  $\Gamma_\theta = \Gamma_{\theta_1}$ , or
2.  $\Gamma_\theta$  and  $\Gamma_{\theta_1}$  are isomorphic, or

3.  $\mathbb{C}/\Gamma_\theta$  is definably isomorphic to  $\mathbb{C}/\Gamma_{\theta_1}$ , more precisely there is a definable one-to-one correspondence between the cosets of  $\Gamma_\theta$  and the cosets of  $\Gamma_{\theta_1}$ .

Hence we *may assume* there are  $a, b, c, d \in \mathbb{Z}$  such that

$$\theta_1 = \frac{a\theta + b}{c\theta + d},$$

thus there is an  $f \in \mathrm{SL}_2(\mathbb{Z})$  such that  $\theta_1 = f(\theta)$ .

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DEPARTMENT OF MATHEMATICAL SCIENCES, TOKAI UNIVERSITY, HIRATSUKA, JAPAN

*E-mail address:* itai@tokai-u.jp